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OPTIMAL IMPULSE CORRECTION UNDER RANDOM PERTURBATIONS

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On the basis of the approach in /1, 2/ we solve the problem of optimal impulse control under random perturbations. We assume that the control resources are limited; control performance is evaluated by a desired final state of the system. We examine both the case of multiple corrections, when there is no restriction on the number of control impulses, as well as the case of single correction. The synthesis problem is reduced to a boundary-value problem with a free boundary for a parabolic equation (the Bellman equation). The number of independent variables is decreased by picking out classes of group-invariant solutions. A further solution is effected by the finite-difference method, using asymptotic expansions. We present the results of calculations. We have found certain exact analytic solutions. Problems of impulse correction under measurement errors were considered earlier in /3, 4/. A method using quasi-variational inequalities has been proposed in /5, 6/ for solving the impulse control problems arising in inventory control theory.

1. Statement of the problem. Let the equations of motion have the form

$$x_k^{\cdot} = a\,(t)\,u_k + b\,(t)\,\xi_k, \quad x_k\,(t^\circ) = x_k^\circ, \quad k = 1, 2, \ldots, n \qquad (1.1)$$

Here t is time, $t^\circ \leqslant t \leqslant T$, x_k and u_k are the components of the phase vector and the control vector, ξ_k are independent white noises of unit intensity, t° and $x^\circ = (x_1^\circ, x_2^\circ, \ldots, x_n^\circ)$ are the initial data. The functions $a\,(t) \geqslant 0$ and $b\,(t) \geqslant 0$ characterize the control effectiveness and the intensity of the random perturbations, respectively. We are required to find the control satisfying the constraint

$$\int_{t^\circ}^{T} \Big(\sum_{j=1}^{n} u_j^2\Big)^{1/2} dt \leqslant q^\circ, \quad q^\circ \geqslant 0 \qquad (1.2)$$

and minimizing the mean of a function of the final state's radius vector

$$I = \langle F\,(r\,(T))\rangle, \quad r = \Big(\sum_{i=1}^{n} x_i^2\Big)^{1/2} \qquad 1.3)$$

The function $F\,(r)$ possesses the properties

$$F\,(r) \geqslant 0, \quad F'\,(r) > 0 \quad (r > 0) \qquad (1.4)$$

Here the prime denotes the derivative with respect to r. Problem (1.1) – (1.3) is solved in two cases: in the absence of restrictions on the number of possible control impulses (multiple correction) and under the assumption of only one impulse (single correction).

We introduce a variable q, equal to unexpended control resources, by the relation

$$q = -\,(u_1^2 + u_2^2 + \ldots + u_n^2)^{1/2}, \quad q\,(t^\circ) = q^\circ \qquad (1.5)$$

We seek the control u in the synthesis form as a function of arbitrary initial data t°, x°, q°. We remark the optimal control problem for the system

$$y_k^{\cdot\cdot} = a(t) u_k + b(t) \xi_k, \quad k = 1, 2, \ldots, n$$

in which a, b, u_k, ξ_k have the same meaning as in (1.1) and which describes the motion of a dynamic system under the action of controls and random perturbations, reduces to problem (1.1)−(1.3) by the change of variables $x_k = y_k + (T - t) y^{\cdot}_k$, $k = 1, 2, \ldots, n$.

2. Basic equations. We consider the multiple correction problem. By $S(t, x, q)$ we denote the minimal value that functional (1.3) can achieve in problem (1.1), (1.3), (1.5) under the initial conditions $t^\circ = t$, $x^\circ = x$, $q^\circ = q$. From the statement of problem (1.1), (1.3), (1.5) it follows that function S is invariant relative to rotation transformations in the space of $x_1, x_2, \ldots, x_n$. This permits us to treat the function S as a function of the three variables

$$\tau = \int_t^T b^2(\lambda) d\lambda, \quad r = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2}, \ q \tag{2.1}$$

The variable τ has been introduced to simplify the subsequent relations. Function $S(\tau, r, q)$ is defined in the domain $D = \{\tau, r, q\colon \tau \geqslant 0, r \geqslant 0, q \geqslant 0\}$ and in this region satisfies the Bellman equation which can be derived by the same method as in Sect. 3 of /1/. As a result we get that the form of the equation depends upon the value of the function

$$Q = a_*(\tau) S_r + S_q \tag{2.2}$$

The value of $Q \leqslant 0$ everywhere in D. The $a_*(\tau)$ in (2.2) denotes the function obtained from $a(t)$ as a result of substitution (2.1). As in /1, 2/ it turns out in the domain D_1, which is defined by the condition $Q < 0$, the function S satisfies the equation

$$S_\tau = \frac{1}{2}\left[S_{rr} + \frac{n-1}{r} S_r\right] \tag{2.3}$$

In the domain $D_2 = D \setminus D_1$ we have $Q = 0$ and, consequently,

$$S(\tau, r, q) = R(r - a_*(\tau) q, \tau) \tag{2.4}$$

(R is an arbitrary function of two variables, yet to be defined). The domains D_1 and D_2 have the following meaning. An uncontrolled motion subject to random perturbations takes place in D_1. An impulse correction is made in D_2, by whose action the phase point (τ, r, q) is shifted along a characteristic of the equation $Q = 0$ and either it shows up on the boundary Γ of domains D_1 and D_2 or the control resources are exhausted. The determination of the boundary Γ solves the optimal control synthesis problem.

The following boundary conditions on the boundaries of domain D exist for the function $S(\tau, r, q)$, which is a solution of Eqs. (2.3) and (2.4) in domains D_1, D_2. From (1.3) follows

$$S(0, r, q) = F(r) \tag{2.5}$$

The condition

$$S_r(\tau, r, q)\big|_{r=0} = 0 \tag{2.6}$$

follows from the symmetry properties of problem (1.1), (1.3), (1.5). When $q = 0$ the function $S(\tau, r, 0) = S^\circ(\tau, r)$ corresponds to uncontrolled motion and satisfies the boundary-value problem

$$S_\tau^\circ - \frac{1}{2}\left[S_{rr}^\circ + \frac{n-1}{r} S_r^\circ\right] = 0, \quad S^\circ(0,r) = F(r), \quad S_r^\circ(\tau,0) = 0 \tag{2.7}$$

In certain cases the solution of this problem can be written out in analytic form. For example, for $F(r) = r^2$ we have

$$S(\tau, r, 0) = S^\circ(\tau, r) = r^2 + n\tau \tag{2.8}$$

The smoothness condition for function Q

$$Q\Big|_\Gamma = 0, \quad \frac{\partial}{\partial r} Q\Big|_\Gamma = 0 \tag{2.9}$$

should be fulfilled on Γ /1/. Thus, the solving of the synthesis problem is reduced to the seeking of the position of boundary Γ by solving a boundary-value problem for Eq. (2.3) with conditions (2.5), (2.6), (2.8), (2.9) in domain D_1.

We pass to the single correction case. By $S^q(\tau, r)$ we denote the minimal value of functional (1.3) in problem (1.1), (1.2) under the condition that only one corrective impulse is possible. Up to the instant of correction the system is subjected to the external random perturbations alone. As shown earlier, the equation

$$S_\tau^q - \frac{1}{2}\left[S_{rr}^q + \frac{n-1}{r} S_r^q\right] = 0 \tag{2.10}$$

for S^q is valid in an appropriate domain δ_1 of variables τ, r, q. The function S^q must satisfy boundary conditions analogous to (2.5), (2.6)

$$S^q(0, r) = F(r), \quad \dot{S}_r^q(\tau, 0) = 0 \tag{2.11}$$

In addition, when $q = 0$, S^q must coincide with the function $S^\circ(\tau, r)$ from (2.7). To derive the conditions to be imposed on function S^q at the correction instant, we note that here the variable r is instantaneously changed by an amount $a_*(\tau)\varkappa$, where $\varkappa$ is the amount of resource expended.

Let us show that $\varkappa = q$ always for an optimal correction. By r^+ and r^- we denote the value of r before and after correction. It is natural to examine only those $\varkappa$ for which $r^- - a_*(\tau)\varkappa \geqslant 0$. Further, let $\varkappa < q$. Two cases can arise after the correction: either $r^+ > 0$ or $r^+ = 0$, i.e. total compensation holds. In both cases the correction proves to be nonoptimal. In the first case $(r^+ > 0)$ we increase the magnitude of the corrective impulse $\varkappa_1$ so as to fulfill the conditions $\varkappa < \varkappa_1 \leqslant q$ and $r_1^+ = r^- - a_*(\tau)\varkappa_1 \geqslant 0$. Since $a_*(\tau) > 0$, we obtain $r_1^+ < r^+$ and after correction the uncontrolled motion starts with a lesser value of r. Therefore, at the end of the process, when $\tau = 0$, the value of functional (1.3) is smaller under correction with impulse $\varkappa_1$ than with impulse $\varkappa$ and the correction being examined is not optimal. In the second case $(r^+ = 0)$ the correction can be applied later, i.e. for a smaller value $\tau_2 < \tau$ of reverse time. We choose the instant τ_2 such that total compensation is possible at instant τ_2, i.e. so as to fulfill the conditions $r_2^+ = r^-(\tau_2) - a_*(\tau_2)\varkappa_2 = 0$, $\varkappa_2 \leqslant q$. This can be done by a choice of τ_2 and $\varkappa_2$ because of the continuity of $a_*(\tau)$ and because of the fact total compensation $(r^+ = 0)$ is possible at the instant τ with an incomplete expenditure of resources $(\varkappa < q)$. After correction at instant τ_2 the uncontrolled motion, just as when correcting at instant τ, starts from the origin $(r_2^+ = 0)$. But since $\tau_2 < \tau$, the time of uncontrolled motion is decreased and, therefore, the functional (1.3) to be minimized is lesser in the case of correction at instant τ_2. Thus, in both cases the assumption $\varkappa < q$ contradicts the optimality of the correction, and $\varkappa = q$ for optimal correction. This fact can be proved with the aid of the maximum principle for parabolic equations.

As a result of correction r is changed to $r - a_*(\tau)\, q$, while since the motion is uncontrolled after correction, the function $S^q(\tau, q)$ coincides with $S^\circ(\tau, r - a_*(\tau)\, q)$ at the instant of correction, where S° is defined by relations (2.7). Thus, on the boundary γ of the domain δ_1 of unclontrolled motion we have the boundary condition

$$S^q(\tau, r)\,|_\gamma = S^\circ(\tau, \quad r - a_*(\tau)\, q) \tag{2.12}$$

The second boundary condition on γ is analogous to (2.9)' and is

$$S_r{}^q(\tau, r)\,|_\gamma = S_r{}^\circ(\tau, r - a_*(\tau)\, q) \tag{2.13}$$

From the definition of domain δ_1 it follows that if the motion starts in $\delta_2 = \delta \setminus \delta_1$, where $\delta = \{\tau, \ r\colon \ \tau \geqslant 0, \ r \geqslant 0\}$, then correction is applied at the initial instant. In domain δ_2 the function S^q is defined by the equality

$$S^q(\tau, r) = S^\circ(\tau, \quad r - a_*(\tau)\, q) \tag{2.14}$$

In summary we conclude that the optimal control in the problem under consideration is characterized by the following factors: it is always impulsive; the direction of the impulse is opposite to the direction of the phase radius vector; the impulse correction corresponds in the phase space (τ, r, q) to a motion along the characterisitics of Eq. (2.9) toward the side of decreasing q. In multiple correction the magnitude of the impulse is determined by the value $q_* \leqslant q$ necessary for hitting, from a given point τ, r, q of domain D_2, onto the boundary Γ along the characteristic of Eq. (2.9), passing through this point; if such an encounter is impossible, then all control resources are spent. In the single correction case the magnitude of the impulse always equals the whole control resource q.

3. Selfsimilar variables and certain exact solutions. Here and later on we assume that the functions $F(r)$, $a(t)$, $b(t)$ have the form

$$F(r) = r^2, \quad a(t) = A_1(T - t)^\alpha, \quad b(t) = B_1(T - t)^\beta \tag{3.1}$$

where A_1, B_1, α, β are given constants. In this case, according to the formulas in Sect. 2, we obtain $a_*(\tau) = A\tau^p$, where $p = \alpha / (1 + 2\beta)$, A is a constant. Under these conditions the boundary-value problems in Sect. 2 are invariant relative to the following one-parameter group of transformations:

$$r \to Cr, \quad \tau \to C^2\tau, \quad q \to C^{-p}q, \quad S \to C^2 S, \quad p = \alpha / (1 + 2\beta) \tag{3.2}$$

where C is a constant. Consequntly, these boundary-value problems have selfsimilar solutions invariant relative to group (3.2). For $p \geqslant {}^1/_2$ we seek such solutions in the form

$$S(\tau, r, q) = \tau W(y, z), \quad S^q(\tau, r) = \tau V(y, z) \tag{3.3}$$
$$y = Aq\tau^{p - 1/2}, \quad z = r\tau^{-1/2}$$

The boundary-vaue problem (2.2), (2.5), (2.6), (2.8), (2.9) in the domain D_1 of variables τ. r, q is transformed under substitution (3.3) into a boundary-value problem for the function $W(y, z)$ in a domain $D_1{}^\circ$ with boundary Γ°. The corresponding equation and boundary conditions are

$$W_{zz} + \left(z + \frac{n-1}{z}\right) W_z - 2W = (2p-1)\, yW_y \tag{3.4}$$

$$W(y, z) \to z^2, \quad z \to \infty, \quad W_z(y, 0) = 0, \quad W(0, z) = z^2 + n \tag{3.5}$$

$$W_z + W_y |_{\Gamma^\circ} = 0, \quad W_{zz} + W_{zy} |_{\Gamma^\circ} = 0 \tag{3.6}$$

In the domain D_2°, into which D_2 is mapped, the function W satisfies the equation

$$W_z + W_y = 0 \tag{3.7}$$

In domain δ_1 the boundary-value problem (2.10) – (2.13) yields, under transformation (3.3), a boundary-value problem in the new variables y, z in a domain δ_1° with boundary γ°

$$V_{zz} + \left(z + \frac{n-1}{z}\right) V_z - 2V = (2p-1)\, yV_y \tag{3.8}$$

$$V(y, z) \to z^2, z \to \infty, \quad V_z(y, 0) = 0, \quad V(0, z) = z^2 + n \tag{3.9}$$

$$V |_{\gamma^\circ} = (z-y)^2 + n, \quad V_z |_{\gamma^\circ} = 2(z-y) \tag{3.10}$$

The domain δ_2 goes over into δ_2° in which, as follows from (2.14), the function V is determined by the finite formula

$$V(y, z) = (z-y)^2 + n \tag{3.11}$$

For $p = 1/2$ the boundary-value problems (3.4) – (3.6) and (3.8) – (3.10) have exact analytic solutions /1/. For the first of these problems we shall seek the solution in the form $W(y, z) = (z^2 + n)\, \Phi(y)$. Thus, the function W constructed satisfies Eq. (3.4) and the boundary conditions (3.5). On the boundary Γ° we obtain

$$2z\Phi(y) + (z^2 + n)\, \Phi(y) = 0, \quad \Phi(y) + z\Phi(y) = 0 \tag{3.12}$$

For a nontrivial solution to exist the determinant of system (3.12), linear and homogeneous relative to $\Phi(y)$, must equal zero. Hence it follows that Γ° is given by the equation $z = \sqrt{n}$, while $\Phi(y) = C \exp(-y / \sqrt{n})$ with an arbitrary constant C. From (3.5) we find that $C = 1$. In the domain $D_2^\circ = \{y, z : z > \sqrt{n}\}$ the solution $W(y, z)$ of Eq. (3.7) is uniquely determined with the aid of the method of characteristics since the values of $W(y, z)$ on Γ° and on the set $y = 0$ are known. In summary, the function $W(y, z)$ is determined by the equalities

$$W(y, z) = \begin{cases} (z^2 + n) \exp(-y / \sqrt{n}), & 0 \leqslant z \leqslant \sqrt{n} \\ 2n \exp((z - y - \sqrt{n}) / \sqrt{n}), & \sqrt{n} \leqslant z \leqslant y + \sqrt{n} \\ (z-y)^2 + n, & y + \sqrt{n} < z \end{cases} \tag{3.13}$$

Figure 1 shows the boundary Γ° of domains D_1° and D_2° for $p = 1/2$ and $n = 3$. The fine lines with arrows mark the trajectories along which motion takes place at the instant of correction. In that part of domain D_2° where $\sqrt{n} \leqslant z \leqslant y + \sqrt{n}$ an impulse correction occurs, leading the phase point onto Γ°, i.e. onto the set $z = \sqrt{n}$; here, not the whole control resource is spent, but that part of it determined by the relation $q_* = y_* = z - \sqrt{n}$ in the part of domain D_2°, where $z > y + \sqrt{n}$, the correction results in the system hitting onto the set $y = q = 0$; in this case the control resource is totally expended. An uncontrolled motion takes place in domain $D_1^\circ = \{y, z\colon z < \sqrt{n}\}$ under the action of the random forces and of $u = 0$. Since here the control resource does not change, the motion in the (y, z)-plane takes place along the straight line $y = q = \text{const}$. Since $z = r / \sqrt{\tau}$, as τ decreases the phase point is shifted toward the side of increasing z. Correction is applied as soon as the point hits from domain $D_1{}^\circ$ onto the boundary Γ° . Thus, a sliding impulse control mode holds along boundary Γ°, which is typical of correction problems with no restriction on the number of pulses.

For the single correction problem, for $p = {}^1/_2$ we obtain by analogous means the equation of the boundary $z = \gamma^\circ(y) = {}^1/_2\,(y + \sqrt{y^2 + 4n})$ and the expression for function

$$V(y, z) = \begin{cases} \dfrac{1}{4n}(z^2 + n)(\sqrt{y^2 + 4n} - y)^2, & 0 \leqslant z \leqslant \gamma^\circ(y) \\ (z - y)^2 + n, & z \geqslant \gamma^\circ(y) \end{cases} \tag{3.14}$$

In this case the domain $\delta_1{}^\circ$ of uncontrolled motion is given by the inequalities $0 \leqslant z < \gamma^\circ(y)$, while in domain $\delta_2{}^\circ$, where $z \geqslant \gamma^\circ(y)$ an impulse correction is applied, as a result of which the control resource is totally expended and the system goes into the state $y = q = 0$. As follows from a comparison of the established functions W and V and of the boundaries Γ° and γ°, the domain $\delta_1{}^\circ$ of uncontrolled motion is wider than the domain $D_1{}^\circ$ and the inequality $W(y, z) \leqslant V(y, z)$ is fulfilled.

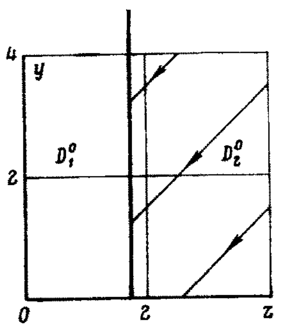

Fig. 1

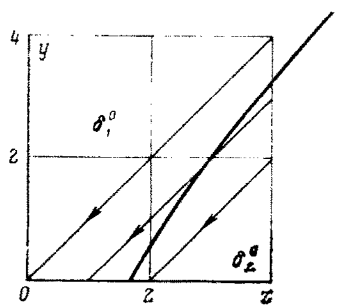

Fig. 2

Figure 2 shows the boundary γ° of domains $\delta_1{}^\circ$ and $\delta_2{}^\circ$ for $n = 3$ and $p = {}^1/_2$. The fine lines with arrows depict the straight lines $y - z = \text{const}$ along which correction takes place from domain $\delta_2{}^\circ$ with the use of the whole control resource. After the whole resource has been used up, motion is continued along the straight line $y = 0$ in both cases of Figs. 1 and 2.

4. Asymptotic expansions. For a numerical solution of boundary-value problems (3.4) – (3.6) and (3.8) – (3.10) we construct, as a preliminary, asymptotic approximations for functions W and V for small y. In the case of multiple correction we seek the solution of Eq. (3.4) in the form

$$W(y, z) = W^\circ(z) + yW^1(z) + y^2 W^2(z) + \ldots \tag{4.1}$$

Here $W^k(z)$ are unknown functions of argument z. In addition we assume that the boundary Γ° is given by a function $\Gamma^\circ(y)$ which admits of the expansion

$$z = \Gamma^\circ(y) = \Gamma_0 + \Gamma_1 y + \Gamma_2 y^2 + \ldots \tag{4.2}$$

(Γ_k are unknown numbers).

Having substituted (4.1) and (4.2) into (3.4) and (3.6) and by picking out terms with like powers of y, we obtain boundary-value problems for ordinary differential equations in the functions $W^k(z)$.

The equation and the boundary conditions for $W^\circ(z)$ are

$$W_{zz}^\circ + \left(z + \frac{n-1}{z}\right) W_z^\circ - 2W^\circ = 0, \quad W_z^\circ(0) = 0, \tag{4.3}$$
$$W^\circ(z) \to z^2, \qquad z \to \infty$$

Hence we find that $W^\circ(z) = z^2 + n$. The function $W^1(z)$ must satisfy the equation

$$W_{zz}^1 + \left(z + \frac{n-1}{z}\right) W_z^1 - (2p+1) W^1 = 0 \tag{4.4}$$

and the boundary condition

$$W_z^1(0) = 0 \tag{4.5}$$

Substituting (4.1) and (4.2) into (3.6) yields two relations

$$2\Gamma_0 + W^1(\Gamma_0) = 0, \quad 2 + W_z^1(\Gamma_0) = 0 \tag{4.6}$$

which are necessary for the simultaneous determination of Γ_0 and $W^1(\Gamma_0)$. The solution of Eq. (4.4) under condition (4.5) can be represented as a series in powers of z, containing an undetermined multiplier k_1. In particular, for $n = 2$ we obtain

$$W^1(z) = k_1\left[1 + \sum_{l=1}^{\infty} \frac{(2p+1)(2p-1)\dots(2p+1-2l+2)}{((2l)!!)^2} z^{2l}\right] = k_1 W^*(z) \tag{4.7}$$

The substitution of (4.7) into the boundary conditions (4.6) yields a transcendental equation in Γ_0

$$W^*(\Gamma_0) = \Gamma_0 W_z^*(\Gamma_0) \tag{4.8}$$

which can be solved numerically. Next, from one of the conditions (4.6) we can find the value of k_1 and, by the same token, completely determine the function $W^1(z)$.

The boundery-value problem for $W^2(z)$ has the form

$$W_{zz}^2 + \left(z + \frac{n-1}{z}\right) W_z^2 - 4pW^2 = 0, \quad W_z^2(0) = 0 \tag{4.9}$$
$$[W_{zz}^\circ(\Gamma_0) + W_z^1(\Gamma_0)]\,\Gamma_1 + W_z^1(\Gamma_0) = -2W^2(\Gamma_0)$$
$$W_{zz}^1(\Gamma_0)\,\Gamma_1 + W_{zz}^1(\Gamma_0) = -2W_z^2(\Gamma_0)$$

The last two boundary conditions in (4.9) contain an unknown quantity Γ_1. Problem (4.9) is solved analogously.

The function $W^\circ(z) + yW^1(z) + y^2W^2(z)$ satisfies Eq. (3.4) to within $O(y^3)$, while on the boundary $z = \Gamma_0 + \Gamma_1 y$ it satisfies conditions (3.6) to within $O(y^2)$, which suffices for the subsequent numerical calculation. A similar procedure for the construction of the asymptotic approximations is used when solving the single correction problem. We note that in the special case of $p = 1$ such an approach was applied in /4/ (*).

*) Riasin, V. A., Optimal strategy of impulse correction under continuous measurements. Preprinty Inst. Prikl. Matem. Akad. Nauk SSSR, Nos. 35, 42.

5. Method and results of numerical solution. Multiple and single correction problems for dimensions $n = 1, 2, 3$ of system (1.1) and for parameter values $p = 1, 2, 4$ were solved numerically in a sufficiently large domain of variables y, z.

The functions W and V at the nodes of the original layer at a distance of a quantity 10^{-2} from the z-axis, as well as the initial positions of boundaries Γ° and γ°, were computed with the aid of the asymptotic expansions to within 10^{-4}. The solution of the parabolic Eqs. (3.4), (3.8) was carried out by the standard implicit finite-difference scheme. The steps Δy and Δz in the variables y and z equalled 0.005. The solution of the difference equations obtained was effected for a fixed y by the sweep method. At each step with respect to y the systems of transcendental Eqs. (3.6) and (3.10) relative to the unknown point of boundary Γ° (or γ°) and the values of the function W (or V) at this point, were solved by Newton's method. As initial approximations we used the quantities obtained at the preceding step. The shifts of position of the boundary Γ° (γ°) at each step in y did not exceed Δz.

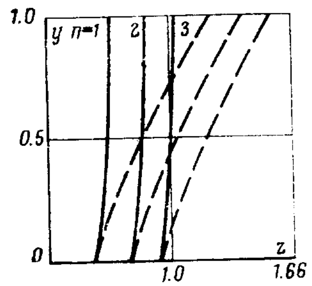

Fig. 3

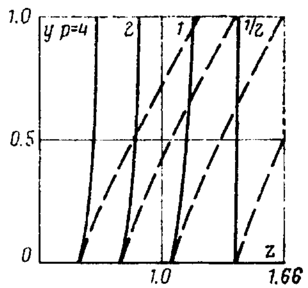

Fig. 4

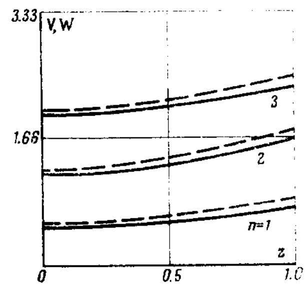

Fig. 5

The results of the calculations are shown in Figs. 3 – 5. Here the solid curves refer to the multiple correction case, while the dashed lines, to the single correction case. Figure 3 shows the boundaries found of domains D_1° and D_2° as well as of δ_1° and δ_2° for $p = 2$ and $n = 1, 2, 3$. The domains D_1°, δ_1° are located to the left, while D_2°, δ_2° to the right of the corresponding boundaries Γ°, γ°. We see that Γ° and γ° merge for $y \to 0$, which, according to (3.3), corresponds to small τ or q. The domain D_1°, as we noted above when comparing the analytic solutions, is in all cases narrower than the domain δ_1°, i.e. the domain of uncontrolled motion is smaller in the case of multiple correction. This corresponds in meaning to the more flexible strategy in multiple correction.

Figure 4 shows the boundaries Γ° and γ° found for $n = 2$ and $p = {}^1/_2, 1, 2, 4$. The solution for $p = {}^1/_2$ was obtained analytically. Figure 5 gives the graphs of the functions $W(y, z)$ and $V(y, z)$ for $p = 1$ and $n = 1, 2, 3$ with a fixed $y = 1$. We see that the inequality $W \leqslant V$, corresponding to the fact that multiple correction leads to a better result, is fulfilled. The values of the performance index for all cases of correction with 2, 3, etc. impulses are contained, obviously, between the corresponding curves on Fig. 5 ($p = 1$), giving the magnitudes of functions W and V for the given dimension n. We also see the fulfillment of the following inequalities:

$$W_1(y,z) < W_2(y,z) < W_3(y,z) \qquad (5.1)$$
$$V_1(y,z) < V_2(y,z) < V_3(y,z)$$

Here the indices $n = 1, 2, 3$ characterize the dimension of system (1.1). Inequalities (5.1) are explained if we interpret the correction problem in case $n = 1$ as the problem of approaching a specified plane in a three-dimensional space at the final instant, the case $n = 2$ as a problem of approaching a straight line, and the case $n = 3$ as a problem of approaching a point. An increase in n signifies an increase in the number of correctable parameters, i.e. a complication of the control problem, and leads to a growth of the functional. Domains D_1°, δ_1° expand as n grows (see Fig. 3).

In conclusion we note that each solution of the correction problem, obtained in the selfsimilar variables y, z, is equivalent to the solution of an entire class of optimal impulse control problems in the original variables τ, r, q.

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OPTIMUM TRANSLATION OF A PENDULUM

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A controlled mechanical system consisting of a suspended load (a pendulum), whose point of suspension can move along a horizontal straight line at some limited speed is considered. The optimum law of motion of the point of suspension is established, which ensures that the pendulum moves over a specified distance in the shortest time and is stationary at the beginning and end of translation, i.e. oscillations are absent at the end point.

This problem arises in investigations of optimum operation conditions of the